

UNCLASSIFIED

AD 402 617

*Reproduced
by the*

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

402 617

63 3 3

402617

DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE 12, RHODE ISLAND

A DUALITY THEOREM FOR A CLASS
OF CONTINUOUS LINEAR
PROGRAMMING PROBLEMS

by

William F. Tyndall

ASTIA
RECEIVED
MAY 1963

TECHNICAL REPORT NO. 1, 1963
Prepared under Contract Nonr-562(15)
for
The Logistics Branch
of the
Office of Naval Research

Abstract of "A Duality Theorem for a Class of
Continuous Linear Programming Problems"

by
William F. Tyndall

Consider the dual pair of continuous linear programming problems defined in the following manner. Let z be a mapping of the closed interval $[0, T]$ of the real line into real euclidean space E^N such that each component function is bounded, measureable, and non-negative. Let w be a similar map into E^M . If B and C are real $M \times N$ matrices and a and c are continuous maps of $[0, T]$ into E^N and E^M respectively, let Z be the set of such functions z satisfying $Bz(t) \leq c(t) + \int_0^t Cz(s)ds$, $0 \leq t \leq T$; and let W be the set of w satisfying $w(t)B \geq a(t) + \int_t^T w(s)Cds$, $0 \leq t \leq T$. Let the primal continuous linear programming problem be: Find $\bar{z} \in Z$ maximizing $\int_0^T z(t) \cdot a(t)dt$, for $z \in Z$; and let its dual problem be: Find $\bar{w} \in W$ minimizing $\int_0^T w(t) \cdot c(t)dt$, for $w \in W$. The following duality theorem is proved.

THEOREM. Hypothesis: I. $\{x \in E^N : Bx \leq 0 \text{ and } x \geq 0\} = \{0\}$, II. B, C , and $c(t)$ have non-negative components, $0 \leq t \leq T$. Conclusion: There exist optimal solutions $\bar{z} \in Z, \bar{w} \in W$. Furthermore, two functions $z \in Z, w \in W$ are optimal if and only if $\int_0^T z(t) \cdot a(t)dt = \int_0^T w(t) \cdot c(t)dt$.

Examples demonstrate that neither hypothesis alone is sufficient.

The economic motivation of these problems is discussed, and the theorem is applied to a dynamic Leontief model of production.

This paper was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Brown University, June 1963. The author wishes to express his gratitude to Professor David Gale for his helpful guidance in the preparation of this paper.

TABLE OF CONTENTS

	Page
1. Preface.....	1
2. Statement of the Duality Theorem.....	2
3. Economic Motivation: An Example.....	4
4. A Discrete Version of the Problem.....	7
5. Some Properties of Duality.....	10
6. Solution of Example 1.....	13
7. On Extending the Duality Theorem.....	15
8. The Main Lemma.....	19
9. Boundedness for the Primal Problem.....	33
10. Effective Boundedness for the Dual Problem.....	38
11. Proof of the Duality Theorem.....	42
12. Related Results.....	43
13. An Economic Application to a Dynamic Leontief Model.....	47
14. Bibliography.....	51

1. PREFACE

The main result of this dissertation is a duality theorem for a class of economically meaningful continuous linear programming problems, a class which is a natural extension of (finite) linear programming problems. The class of problems is defined and the theorem stated in Section 2.

In keeping with the economic motivation of these problems an attempt is made to illustrate by several examples some simple problems of economic significance. In fact, Sections 3, 4, and 6 are devoted primarily to an exposition of this economic background together with a detailed solution of one example which serves to illustrate the usefulness of the concept of duality as a technique for arriving at solutions and proving the results obtained are indeed optimal. The connection between these continuous linear programming problems and finite linear programming problems should become apparent. The final section applies the duality theorem to a dynamic Leontief production model.

The mathematical results are contained in Sections 5 and 7-12. Section 5 contains some results relating the primal and dual problems, while Section 7 demonstrates that a duality theorem for this class of continuous linear programming problems requires hypotheses which are more restrictive than those required in the finite case.

Section 8 contains the proof of the main lemma, which

itself is a duality theorem with some rather restrictive hypotheses. Sections 9 and 10 demonstrate that the hypotheses of the duality theorem ensure that the main lemma is applicable. The proof of the duality theorem is completed in Section 11, while Section 12 contains some related results.

2. STATEMENT OF THE DUALITY THEOREM

Let z be a function mapping the closed interval $[0, T]$ of the real line into E^N , real euclidean space of dimension N . For $t \in [0, T]$ let $z(t) = (\xi_1(t), \dots, \xi_N(t))$. We shall assume that ξ_j is a bounded, measureable (with respect to Lebesgue measure of the real line) function for $j = 1, \dots, N$, and call such a function z bounded and measureable.

Let a and c be continuous functions mapping $[0, T]$ into E^N and E^M , respectively, and let B and C be real $M \times N$ matrices. We use the following notation: For $d = (\delta_1, \dots, \delta_N) \in E^N$, $e = (\epsilon_1, \dots, \epsilon_M) \in E^M$, and for A any real $M \times N$ matrix, let eA and Ad denote the suitable vector-matrix products. (Note that we do not use the familiar notation Ad^T .) The inner product of $d, d' \in E^N$ is given by $d \cdot d' = \sum_{j=1}^N \delta_j \delta'_j$. Finally, we say $d \leq d'$ if and only if $\delta_j \leq \delta'_j$ for $j = 1, \dots, N$.

Define Z to be the set of all bounded, measureable functions $z : [0, T] \longrightarrow E^N$ such that $z(t) \geq 0$ and

$Bz(t) \leq c(t) + \int_0^t Cz(s)ds$ for $0 \leq t \leq T$. Similarly, define W to be the set of all bounded, measurable functions $w : [0, T] \rightarrow E^M$ such that $w(t) \geq 0$ and $w(t)B \geq a(t) + \int_t^T w(s)C ds$ for $0 \leq t \leq T$. A function $z \in Z$ or $w \in W$ will be called feasible. We shall assume that neither B nor C is a zero matrix.

The continuous linear programming problem to be discussed can now be stated.

PRIMAL PROBLEM. Find some $\bar{z} \in Z$ such that

$$\int_0^T \bar{z}(t) \cdot a(t)dt = \max \left\{ \int_0^T z(t) \cdot a(t)dt : z \in Z \right\}.$$

This problem is called the primal problem to distinguish it from the related dual problem.

DUAL PROBLEM. Find some $\bar{w} \in W$ such that

$$\int_0^T \bar{w}(t) \cdot c(t)dt = \min \left\{ \int_0^T w(t) \cdot c(t)dt : w \in W \right\}.$$

Observe that if C were a zero matrix, contrary to our assumption, and if a and c were constant, then we are left with a standard maximum problem and its dual, well-known objects in the theory of linear programming. (See [1], for example.) It is reasonable, therefore, to consider this class of continuous linear programming problems as an extension of a class of (finite) linear programming problems and to attempt

to extend the basic theorems of linear programming to this larger class.

Indeed, the main result of this thesis is an analogue of the duality theorem valid for a class of economically meaningful continuous linear programming problems.

THEOREM 1.

Hypothesis:

- I. $\{x \in E^N : Bx \leq 0 \text{ and } x \geq 0\} = \{0\}$
- II. $B, C,$ and $c(t)$ have non-negative components for $0 \leq t \leq T$.

Conclusion: There exist optimal solutions $\bar{z} \in Z$ and $\bar{w} \in W$. Furthermore, two feasible functions z and w are optimal if and only if $\int_0^T z(t) \cdot a(t) dt = \int_0^T w(t) \cdot c(t) dt$.

Theorems 5 and 6, p. 16, demonstrate that neither hypothesis alone is sufficient.

It should be noted that the condition that both primal and dual problems be feasible, a sufficient condition for the duality theorem of (finite) linear programming, is no longer sufficient to yield a duality theorem for these continuous linear programming problems. This is demonstrated by Theorem 4, p. 16.

3. ECONOMIC MOTIVATION: AN EXAMPLE

The continuous linear programming problem which has been defined is called a "bottleneck problem" by Richard Bellman.

In [2] he examines several economically meaningful examples and discusses the dual problem, using properties of the dual to obtain solutions to his examples. Furthermore, Dorfman, Samuelson, and Solow in [3] discuss a discrete version of this problem in their examination of a dynamic Leontief system.

In order both to illustrate the economic background and to give an illustration of the practical usefulness of the properties of duality, we shall look at a simple example, a modification of one due to Philip Wolfe [4].

EXAMPLE 1. Let us imagine a steel manufacturer who is faced with the choice of allocating part of his steel output at a rate ξ_1 to build a larger steel factory, thereby enabling steel to be produced more rapidly, and, on the other hand, allocating the remainder at a rate ξ_2 to his stockpile of steel, the ultimate goal of his production effort. Assume that there are γ_1 units of steel capacity initially, and the units are chosen so that one unit of steel capacity enables steel to be produced at the rate of one unit of steel per unit time.

The rate of production of steel, $\xi_1 + \xi_2$, is assumed to depend linearly on the capacity of the factory. Moreover, it is assumed that the value of the steel stockpile depends linearly on the quantity of steel available, so that the manufacturer seeks to maximize his stockpile of steel by the end of T units of time.

At each time t , $0 \leq t \leq T$, the manufacturer must determine the rates of allocation, $\xi_1(t)$ and $\xi_2(t)$, both of which are to be non-negative (i.e., no "scrapping" is allowed).

Now the amount of steel capacity at time t is just $\gamma_1 + \int_0^t \xi_1(s)ds$. The constraint that limited factory capacity imposes upon the rate of production $\xi_1(t) + \xi_2(t)$ is

$$(3.1) \quad \xi_1(t) + \xi_2(t) \leq \gamma_1 + \int_0^t \xi_1(s)ds, \quad 0 \leq t \leq T.$$

We impose the reasonable technological constraint

$$(3.2) \quad \xi_1(t) \leq \gamma_2, \quad 0 \leq t \leq T$$

which just says that the factory cannot be enlarged arbitrarily in a limited amount of time.

Given a technology constrained by (3.1) and (3.2), the manufacturer is faced with the problem: Find non-negative functions ξ_1, ξ_2 maximizing $\int_0^T \xi_2(t)dt$ subject to (3.1) and (3.2).

If we let $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $a(t) \equiv (0, 1)$, $c(t) \equiv (\gamma_1, \gamma_2)$, and $z(t) = (\xi_1(t), \xi_2(t))$, the problem can be stated: Find a (bounded, measurable) function $z : [0, T] \longrightarrow E^2$ with $z(t) \geq 0$ and $Bz(t) \leq c(t) + \int_0^t Cz(s)ds$, for $0 \leq t \leq T$, maximizing $\int_0^T z(t) \cdot a(t)dt$.

In order to examine in detail just one of the several possible cases which arise, we shall assume that

$$(3.3) \quad 0 < \gamma_1 < \gamma_2 \quad \text{and} \quad T > 1 + \ln \frac{\gamma_2}{\gamma_1} .$$

It will be seen later and might be argued now from a commonsense point of view that an optimal allocation policy will consist of building steel capacity initially (at the expense of not increasing the steel stockpile) and toward the end of the production interval using this enlarged capacity to produce steel exclusively for the stockpile at the larger rate then permissible. Anticipating this, we set $\xi_2 \equiv 0$ in (3.1) and solve $\xi_1(t) = \gamma_1 + \int_0^t \xi_1(s)ds$ to get $\xi_1(t) = \gamma_1 e^t$. This policy is feasible so long as $\gamma_1 e^t \leq \gamma_2$. When $t = t_1 \equiv \ln \gamma_2 / \gamma_1$, a "bottleneck" develops; one cannot continue to build steel capacity at the rate $\gamma_1 e^t$ for $t > t_1$ since (3.2) would be violated. As a result of this factory building bottleneck a new policy must be adopted for $t > t_1$.

We postpone the solution of this example until after we have proved some useful results extending some theorems of linear programming relating the primal and dual problems. Then we will solve concurrently both the primal and dual problems for this example, thereby illustrating the usefulness of the dual problem, both in arriving at a solution to the primal problem and in proving that the result so obtained is indeed the optimal one.

4. A DISCRETE VERSION OF THE PROBLEM

In order to motivate the definition of the dual problem

we shall examine a discrete version of the continuous problem. This concept, expounded in [2], will be used later in proving the duality theorem.

Suppose that the interval $[0, T]$ is divided into n equal parts of length $\Delta t^n = T/n$. If $t_k^n = kT/n$, $k = 0, \dots, n$, let us reformulate the example of Section 3. We shall now regard $\xi_1(t_k^n)$, for example, as the (constant) rate at which steel is used to enlarge the factory during the time interval $[t_k^n, t_{k+1}^n)$. During this interval, therefore, the steel capacity is enlarged by $\xi_1(t_k^n)\Delta t^n$. The constraint (3.1) now becomes

$$(4.1) \quad \begin{aligned} &\xi_1(t_0^n) + \xi_2(t_0^n) \leq \gamma_1, \quad \text{and} \\ &\xi_1(t_k^n) + \xi_2(t_k^n) \leq \gamma_1 + \Delta t^n \sum_{v=0}^{k-1} \xi_1(t_v^n), \quad k = 1, \dots, n \end{aligned}$$

while (3.2) becomes

$$(4.2) \quad \xi_1(t_k^n) \leq \gamma_2, \quad k = 0, \dots, n.$$

We thus seek $\xi_j(t_k^n) \geq 0$ for $j = 1, 2; k = 0, \dots, n$, maximizing $\sum_{k=0}^n \xi_2(t_k^n)$, thereby maximizing $\Delta t^n \sum_{k=0}^n \xi_2(t_k^n)$, subject to (4.1) and (4.2). We remark that $z(t_n^n)$ is sought for purely formal reasons. The introduction of this variable yields a more useful formulation for the discrete dual problem.

In vector-matrix notation this reads: Find non-negative

vectors $z(t_k^n) \in E^2$, $k = 0, \dots, n$, maximizing
 $\sum_{k=0}^n z(t_k^n) \cdot a(t_k^n)$ subject to

$$(4.3) \quad \begin{aligned} Bz(t_0^n) &\leq c(t_0^n) \\ Bz(t_k^n) &\leq c(t_k^n) + \Delta t^n \sum_{v=0}^{k-1} Cz(t_v^n), \quad k = 1, \dots, n \end{aligned}$$

This discrete version of the problem is just an example of a finite, if overblown, standard maximum problem. To this linear programming problem is paired a standard minimum problem, its dual problem. It is: Find non-negative vectors $w(t_k^n) \in E^2$, $k = 0, \dots, n$, minimizing $\sum_{k=0}^n w(t_k^n) \cdot c(t_k^n)$, subject to

$$(4.4) \quad \begin{aligned} w(t_n^n)B &\geq a(t_n^n) \\ w(t_k^n)B &\geq a(t_k^n) + \Delta t^n \sum_{v=k+1}^n w(t_v^n)C, \quad k = 0, \dots, n-1 \end{aligned}$$

It is informative to compare this discrete dual problem with the continuous dual problem. We see in the comparison that the dual problem associated to the discrete version of the continuous primal problem is itself a discrete version of the dual problem associated with the continuous primal problem.

5. SOME PROPERTIES OF DUALITY

In this section we extend some of the theorems of linear programming to continuous linear programming problems. These results are proved in [2], but are included here for the sake of completeness.

For any bounded, measureable function $z : [0, T] \longrightarrow E^N$, let us define the function $\ell(z) : [0, T] \longrightarrow E^M$ by $\ell(z)(t) = Bz(t) - \int_0^t Cz(s)ds$, for $t \in [0, T]$. It is clear that $\ell(z)$ is again a bounded, measureable function.

Similarly, for any bounded, measureable function $w : [0, T] \longrightarrow E^M$, we define the (bounded, measureable) function $\ell^*(w) : [0, T] \longrightarrow E^N$ by $\ell^*(w)(t) = w(t)B - \int_t^T w(s)Cds$ for $t \in [0, T]$.

LEMMA 1. ℓ^* is "adjoint" to ℓ in the sense that

$$\int_0^T \ell(z)(t) \cdot w(t)dt = \int_0^T \ell^*(w)(t) \cdot z(t)dt .$$

PROOF. Clearly both integrals are defined and finite. By expanding the products in the integrands we see that it suffices to prove that $\int_0^T (\omega_i(t) \int_0^t \xi_j(s)ds)dt = \int_0^T (\xi_j(t) \int_t^T \omega_i(s)ds)dt$ for $i = 1, \dots, M$; $j = 1, \dots, N$. We use integration by parts, letting $u(t) = \int_0^t \xi_j(s)ds$ and $v(t) = \int_t^T \omega_i(s)ds$. Note that $\int_0^T (\omega_i(t) \int_0^t \xi_j(s)ds)dt = - \int_0^T u(t)v'(t)dt = -u(t)v(t)]_0^T + \int_0^T v(t)u'(t)dt = \int_0^T (\xi_j(t) \int_t^T \omega_i(s)ds)dt$.

LEMMA 2. If $z \in Z$ and $w \in W$, then $\int_0^T z(t) \cdot a(t) dt \leq \int_0^T w(t) \cdot c(t) dt$.

PROOF. By the hypothesis $l(z)(t) \leq c(t)$ and $l^*(w)(t) \geq a(t)$, $0 \leq t \leq T$. Since $z(t)$ and $w(t)$ are non-negative, $l(z)(t) \cdot w(t) \leq w(t) \cdot c(t)$ and $l^*(w)(t) \cdot z(t) \geq z(t) \cdot a(t)$, $0 \leq t \leq T$. The conclusion follows by integrating and using Lemma 1.

This lemma has an immediate corollary.

COROLLARY. When the quantities exist

$$\sup \left\{ \int_0^T z(t) \cdot a(t) dt : z \in Z \right\} \leq \inf \left\{ \int_0^T w(t) \cdot c(t) dt : w \in W \right\}.$$

THEOREM 2 (Optimality Condition). If there exist functions $\bar{z} \in Z$ and $\bar{w} \in W$ such that $\int_0^T \bar{z}(t) \cdot a(t) dt = \int_0^T \bar{w}(t) \cdot c(t) dt$, then \bar{z} and \bar{w} are the optimal solutions of their respective problems.

PROOF. Using Lemma 2 note that for all $z \in Z$, $\int_0^T z(t) \cdot a(t) dt \leq \int_0^T \bar{w}(t) \cdot c(t) dt = \int_0^T \bar{z}(t) \cdot a(t) dt$, and for all $w \in W$, $\int_0^T w(t) \cdot c(t) dt \geq \int_0^T \bar{z}(t) \cdot a(t) dt = \int_0^T \bar{w}(t) \cdot c(t) dt$.

Let us denote the i^{th} component of $Bz(t) \in E^M$, for example, by $(Bz(t))_1$.

THEOREM 3 (Equilibrium Conditions). Let $z \in Z$ and $w \in W$. Then $\int_0^T z(t) \cdot a(t) dt = \int_0^T w(t) \cdot c(t) dt$ if and only if both (i) and (ii) are satisfied for almost all $t \in [0, T]$.

(i) For $i = 1, \dots, M$, $(Bz(t))_i < \gamma_i(t) + (\int_0^t Cz(s) ds)_i$
implies $\omega_i(t) = 0$.

(ii) For $j = 1, \dots, N$, $(w(t)B)_j > \alpha_j(t) + (\int_t^T w(s)Cs ds)_j$
implies $\zeta_j(t) = 0$.

PROOF. Suppose the conditions (i) and (ii) are satisfied almost everywhere in $[0, T]$. Multiply the i^{th} inequality by $\omega_i(t)$ and the j^{th} inequality by $\zeta_j(t)$ and sum to get the equalities

$$(5.1) \quad \sum_{i=1}^M \omega_i(t) (Bz(t))_i = \sum_{i=1}^M \omega_i(t) \gamma_i(t) + \sum_{i=1}^M \omega_i(t) \left(\int_0^t Cz(s) ds \right)_i$$

$$(5.2) \quad \sum_{j=1}^N \zeta_j(t) (w(t)B)_j = \sum_{j=1}^N \zeta_j(t) \alpha_j(t) + \sum_{j=1}^N \zeta_j(t) \left(\int_t^T w(s)Cs ds \right)_j$$

which hold for almost all $t \in [0, T]$. The equality $\int_0^T z(t) \cdot a(t) dt = \int_0^T w(t) \cdot c(t) dt$ then follows by integrating and using Lemma 1.

Conversely, if the conditions fail for t in some subset of $[0, T]$ having positive measure, the equality in (5.1) or (5.2) must be replaced by strict inequality for t in a set of positive measure. Consequently, $\int_0^T z(t) \cdot a(t) dt < \int_0^T w(t) \cdot c(t) dt$. This completes the proof.

6. SOLUTION OF EXAMPLE 1

We now return to the example of Section 3. We shall use the equilibrium conditions to arrive at a solution.

The dual problem is defined to be: Find non-negative functions ω_1 and ω_2 minimizing $\int_0^T (\gamma_1 \omega_1(t) + \gamma_2 \omega_2(t)) dt$ subject to

$$(6.1) \quad \omega_1(t) + \omega_2(t) \geq \int_t^T \omega_1(s) ds, \quad 0 \leq t \leq T$$

$$(6.2) \quad \omega_1(t) \geq 1, \quad 0 \leq t \leq T.$$

We saw that for $0 \leq t \leq t_1$, $\xi_1(t) = \gamma_1 e^t$ and $\xi_2(t) = 0$ satisfy the constraints (3.1) and (3.2). Now at the end of the time interval $[0, T]$ we suspect that an optimal policy would stockpile all of the steel produced. We thus want to set $\xi_1(t) = 0$ and let ξ_2 satisfy (3.1) as an equality for $t_2 \leq t \leq T$, where t_2 , $t_1 \leq t_2 < T$, is to be determined.

To do this we now look at the dual problem. If $\xi_1 \equiv 0$ for $t_2 \leq t \leq T$, (3.2) is a strict inequality, and hence,

from the equilibrium conditions (Theorem 3) the dual variable ω_2 corresponding to the relation (3.2) must vanish* in this interval in order for ω_2 to be part of an optimal policy. Furthermore, since it is desired that ζ_2 be positive, the corresponding dual relation (6.2) must be an equality for $t_2 \leq t \leq T$. In this interval we thus desire that $\omega_1(t) \equiv 1$ and $1 \geq \int_t^T 1 dt$. This will be true if $t_2 = T - 1$. But by the assumption (3.3), p. 7, $T - 1 > \ln \gamma_2 / \gamma_1 \equiv t_1$, so $w(t) = (1, 0)$ for $t_2 \leq t \leq T$ is the function desired.

To complete our solution to the primal problem we see that for $t_1 < t < t_2$ (3.1) and (3.2) can be solved as equalities by taking $\zeta_1(t) = \gamma_2$ and $\zeta_2(t) = (t - t_1)\gamma_2$.

The equilibrium conditions now require that in (t_1, t_2) the dual functions ω_1 and ω_2 must satisfy (6.1) and (6.2) as equalities. Solving for ω_1 and ω_2 we obtain $\omega_1(t) = 1$ and $\omega_2(t) = t_2 - t$, $t_1 < t < t_2$.

Finally we seek a w to pair with z for $0 \leq t \leq t_1$. The equilibrium conditions lead to the equations $\omega_2(t) = 0$ and $\omega_1(t) = \int_t^T \omega_1(s) ds$. Thus let $\omega_1(t) = (T - t_1)e^{t_1 - t}$ and $\omega_2(t) = 0$ for $t \in [0, t_1]$.

Since the functions z and w which we have obtained are feasible and satisfy the equilibrium conditions, we are guaranteed by Theorem 3 that these z and w are the optimal solutions of the problem and its dual. As a check, using the

* The "almost everywhere" will not be repeated in this discussion.

optimality condition of Theorem 2, we note that

$$\begin{aligned}\int_0^T \zeta_2(t) dt &= \frac{\gamma_2}{2} (t_2 - t_1)^2 + \gamma_2 (T - t_1) \\ &= \int_0^T (\gamma_1 \omega_1(t) + \gamma_2 \omega_2(t)) dt .\end{aligned}$$

This example illustrates the complexity already encountered in the 2×2 case. With a greater number of variables and constraints one is forced to turn to more elaborate techniques to obtain a solution. Bellman approaches this problem employing a functional equation. Several examples are solved using this method in [2]. R. Sherman Lehman, leaning more heavily on the linearity of the problem, has devised a "continuous simplex method" to solve some of these problems in [5]. Neither of these treatments is a model of mathematical rigor. Indeed the avowed purpose of these authors is to discover solutions where possible; they are not constrained by a desire for complete rigor in their methods.

7. ON EXTENDING THE DUALITY THEOREM

In Section 5 it was noted that the optimality and equilibrium conditions have been extended without difficulty to this class of continuous linear programming problems. The important duality theorem, on the other hand, does not extend

to this class without modifying the hypothesis.

In this section we discuss two examples which will prove Theorems 4, 5, and 6. These theorems justify the inclusion of Hypotheses I and II in Theorem 1, stated on p. 4.

THEOREM 4. The existence of functions z and w , feasible for their respective problems, is not sufficient to guarantee the existence of optimizing solutions.

PROOF. Example 2, to follow.

THEOREM 5. Hypothesis I without Hypothesis II is not sufficient to guarantee the existence of optimizing solutions.

PROOF. Example 2.

THEOREM 6. Hypothesis II without Hypothesis I is not sufficient to guarantee the existence of optimizing solutions.

PROOF. Example 3, to follow.

EXAMPLE 2. Let γ_1 and γ_2 be positive numbers and assume that $T > \gamma_1/\gamma_2$. Find a non-negative function maximizing $\int_0^T \xi(t) dt$ subject to

$$(7.1) \quad 0 \leq \gamma_1 - \int_0^t \xi(s) ds, \quad 0 \leq t \leq T$$

$$(7.2) \quad \xi(t) \leq \gamma_2, \quad 0 \leq t \leq T.$$

If one imagines ξ to be the rate of production of automobiles, say, and γ_2 to be the (constant) auto capacity available, (7.2) gives a realistic bound on this rate. If γ_1 is the amount of steel initially available, and if units

are chosen so that one unit of steel is required to make one automobile, then (7.1) is the requirement that the amount of steel at any time be non-negative.

The solution of this problem is obvious. Let $\xi(t) = \gamma_1/T$. By the hypothesis $\gamma_1/T < \gamma_2$; also $\int_0^t \xi(s)ds = t\gamma_1/T \leq \gamma_1$, for $0 \leq t \leq T$, so that this ξ is feasible. In view of (7.1) this ξ will produce the maximum value γ_1 .

The dual of this problem is a different matter, however. One seeks non-negative functions ω_1 and ω_2 minimizing $\int_0^T (\gamma_1 \omega_1(t) + \gamma_2 \omega_2(t))dt$ subject to

$$(7.3) \quad \omega_2(t) \geq 1 - \int_t^T \omega_1(s)ds, \quad 0 \leq t \leq T.$$

Now if there are feasible functions ω_1 and ω_2 attaining a minimum γ_1 , they must satisfy the equilibrium conditions when paired with $\xi(t) = \gamma_1/T$ (Theorem 3). Since (7.2) is a strict inequality throughout the interval $[0, T]$, ω_2 must vanish almost everywhere; and since $\xi(t) > 0$, ω_1 must satisfy (7.3) as an equality almost everywhere. We thus seek ω_1 such that $\int_t^T \omega_1(s)ds = 1$ almost everywhere in $[0, T]$. But no measurable function has this property, so we must conclude that the dual problem has no solution, even though the primal problem does.

This dual problem is feasible, however. In fact there

exists a sequence of functions $w^n \in W$ such that

$$\lim_n \int_0^T w^n(t) \cdot c(t) dt = \gamma_1.$$

Define $w^n(t)$ for $n > 1/T$ by

$$w_1^n(t) = \begin{cases} 0, & 0 \leq t \leq T - \frac{1}{n} \\ n, & T - \frac{1}{n} < t \leq T \end{cases}$$

$$w_2^n(t) = \begin{cases} 0, & 0 \leq t \leq T - \frac{1}{n} \\ 1 - (T - t)n, & T - \frac{1}{n} < t \leq T \end{cases}.$$

Now observe that each w^n is feasible, and, moreover,

$$\int_0^T (\gamma_1 w_1^n(t) + \gamma_2 w_2^n(t)) dt = \gamma_1 + \gamma_2/2n \longrightarrow \gamma_1 \text{ as } n \longrightarrow \infty.$$

In this example both primal and dual problems are feasible, so the proof of Theorem 4 is now complete. Furthermore, since the matrix B of Hypothesis I is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, I is satisfied. The matrix $C = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, however, is not non-negative, so Hypothesis II is not satisfied. This proves Theorem 5.

REMARK. Note that $\lim_n w_2^n(t) = 0$ for $0 \leq t < T$.

Looking at w_1^n one might be tempted to say, naively, that " w_1^n approaches a delta function with unit weight at $t = T$." It should be noted that in their attempt to treat a large class of these problems Bellman and Lehman admit delta functions or even derivatives of delta functions in their solutions when the problem or its dual has no measureable

function as a solution ([2] and [5]). From a rigorous point of view one is led to try to reformulate the problem so as to accept a measure as a solution in these cases; but this extension leads to even more difficulties (see [5], p. 21). As formulated here, however, only functions will be admitted as possible solutions.

EXAMPLE 3. Let γ be a positive number. Find non-negative functions ζ_1 and ζ_2 maximizing $\int_0^T \zeta_1(t) dt$ subject to

$$\zeta_1(t) \leq \gamma + \int_0^t \zeta_2(s) ds, \quad 0 \leq t \leq T.$$

By letting $\zeta_2^n(t) = n$ and $\zeta_1^n(t) = \gamma + nt$ for $0 \leq t \leq T$, we find that $\int_0^T \zeta_1^n(t) dt = \gamma T + nT^2/2$, so that there is no maximum.

Here Hypothesis I is not satisfied, while Hypothesis II is. This proves Theorem 6.

8. THE MAIN LEMMA

We turn now to the proofs of some lemmas which will lead to the proof of the duality theorem.

We shall approach the solution of the continuous problem by means of a sequence of finite discrete approximations. This idea was discussed in Section 4.

For $n = 1, 2, \dots$ let $\Delta t^n = T/n$ and let $t_k^n = kT/n$ for $k = 0, \dots, n$.

Let

$$a^n(t_k^n) = \frac{1}{\Delta t^n} \int_{t_k^n}^{t_{k+1}^n} a(t) dt, \quad$$

for $k = 0, \dots, n-1$, and let $a^n(t_n^n) = a^n(t_{n-1}^n)$.

Let

$$c^n(t_k^n) = \frac{1}{\Delta t^n} \int_{t_{k-1}^n}^{t_k^n} c(t) dt,$$

for $k = 1, \dots, n$, and let $c^n(t_0^n) = c^n(t_1^n)$.

We now let P^n be the problem of finding non-negative vectors $z^n(t_0^n), \dots, z^n(t_n^n) \in E^N$ maximizing $\sum_{k=0}^n z^n(t_k^n) \cdot a^n(t_k^n)$ subject to

$$Bz^n(t_0^n) \leq c^n(t_0^n)$$

and

$$Bz^n(t_k^n) \leq c^n(t_k^n) + \Delta t^n \sum_{v=0}^{k-1} Cz^n(t_v^n), \quad k = 1, \dots, n.$$

The dual of this finite linear programming problem seeks non-negative vectors $w^n(t_0^n), \dots, w^n(t_n^n) \in E^M$ minimizing $\sum_{k=0}^n w^n(t_k^n) \cdot c^n(t_k^n)$ subject to

$$w^n(t_n^n)B \geq a^n(t_n^n)$$

and

$$w^n(t_k^n)B \geq a^n(t_k^n) + \Delta t^n \sum_{v=k+1}^n w^n(t_v^n)C, \quad k = 0, \dots, n-1.$$

If $z^n(t_0^n), \dots, z^n(t_n^n)$ are feasible for P^n , define an associated step function $\hat{z}^n : [0, T] \longrightarrow E^N$ by

$$z^n(t) = \begin{cases} z^n(t_0^n), & t_0^n \leq t < t_1^n \\ z^n(t_k^n), & t_k^n \leq t < t_{k+1}^n, \quad k = 1, \dots, n-2 \\ z^n(t_{n-1}^n), & t_{n-1}^n \leq t \leq t_n^n \end{cases}$$

Similarly, if $w^n(t_0^n), \dots, w^n(t_n^n)$ are feasible for the dual of P^n , define the step function $\hat{w}^n : [0, T] \longrightarrow E^M$ by

$$w^n(t) = \begin{cases} w^n(t_1^n), & t_0^n \leq t \leq t_1^n \\ w^n(t_k^n), & t_{k-1}^n < t \leq t_k^n, \quad k = 2, \dots, n-1 \\ w^n(t_n^n), & t_{n-1}^n < t \leq t_n^n \end{cases}$$

We remark that it is not true in general that such associated step functions are feasible for the continuous problem.

In order to prove the main lemma it is necessary to relax temporarily the requirement that inequalities be satisfied for all $t \in [0, T]$ in the definitions of Z and W , the sets of functions feasible for the continuous problem and its dual.

For $p = 1, \infty$, let $L^p[0, T]$ be the family of equivalence classes of real-valued Lebesgue-measurable functions on $[0, T]$ having finite L^p norm. For a map $z : [0, T] \longrightarrow E^N$ we shall say $z \in L_N^p$ if $z_j \in L^p[0, T]$ for $j = 1, \dots, N$.

Define

$$Z_L = \left\{ z \in L_N^1 : Bz(t) \leq c(t) + \int_0^t Cz(s)ds \text{ and } \right. \\ \left. z(t) \geq 0 \text{ for almost all } t \in [0, T] \right\}$$

and let

$$Z_L' = \left\{ z \in L_N^1 : \int_{t_1}^{t_2} Bz(t)dt \leq \int_{t_1}^{t_2} c(t)dt \right. \\ \left. + \int_{t_1}^{t_2} \left(\int_0^t Cz(s)ds \right) dt \text{ for } 0 \leq t_1 < t_2 \leq T \right. \\ \left. \text{and } z(t) \geq 0 \text{ a.e.} \right\}.$$

Similar definitions of W_L and W_L' can be associated to W .

We assert: $Z_L = Z_L'$. Since the function $z(t) \longrightarrow Bz(t) - \int_0^t Cz(s)ds$ maps L_N^1 into L_M^1 , one sees by looking at each component separately that it suffices to prove $Z_L = Z_L'$ for the special case $N = M = 1$.

Now clearly $Z_L \subset Z_L'$. The other inclusion follows from a well-known result in the theory of real functions, namely: If $f \in L^1$ and $F(x) = \int_a^x f(t)dt$, then F' exists almost everywhere and $F'(x) = f(x)$ almost everywhere.

We consider t_1 fixed, $0 \leq t_1 < t_2 \leq T$. Then

$$\begin{aligned} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} Bz(t) dt &\leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} c(t) dt \\ &+ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left(\int_0^t Cz(s) ds \right) dt . \end{aligned}$$

But, as $t_2 \longrightarrow t_1^+$, the left hand side approaches $Bz(t_1)$ and the right hand side approaches $c(t_1) + \int_0^{t_1} Cz(s) ds$ for almost all values of $t_1 \in [0, T]$. Hence $Z_L' \subset Z_L$.

A similar proof shows that $W_L = W_L'$.

We shall need a result from the theory of Banach spaces (see [6], p. 123). The dual space of the separable Banach space $L^1[0, T]$ can be identified with $L^\infty[0, T]$. An important property enjoyed by the dual of a separable Banach space is weak* sequential compactness for sets bounded in the strong topology. Restated for our purposes, this becomes:

Let $\lambda_n \in L^\infty[0, T]$ and assume $\|\lambda_n\|_\infty \leq \rho$ for $n = 1, 2, \dots$. Then there exist $\lambda \in L^\infty[0, T]$ and a subsequence $\{n_k\}$ such that $\lambda_{n_k} \longrightarrow \lambda$ (weak*); that is $\int_0^T \lambda_{n_k} f \longrightarrow \int_0^T \lambda f$, for all $f \in L^1[0, T]$. It is convenient to restate this as

LEMMA 3. Let $\lambda_n \in L^\infty[0, T]$ and assume that $\|\lambda_n\|_\infty$ for $n = 1, 2, \dots$. Then there exist $\lambda \in L^\infty[0, T]$ and a subsequence $\{n_k\}$ such that

$$\int_{t_1}^{t_2} \lambda_{n_k}(t) dt \longrightarrow \int_{t_1}^{t_2} \lambda(t) dt, \text{ for } 0 \leq t_1 < t_2 \leq T .$$

PROOF. Let χ be the characteristic function for $[t_1, t_2]$. Since $[0, T]$ has finite measure, $\chi \in L^1[0, T]$, and the conclusion follows from the result quoted above.

We are now able to prove the main lemma.

LEMMA 4. For $n = 1, 2, \dots$ let P^n be the discrete problem associated to the continuous problem. Assume that for all n there exist vectors $\{z^n(t_0^n), \dots, z^n(t_n^n)\}$ and $\{w^n(t_0^n), \dots, w^n(t_n^n)\}$ solving P^n . Assume, furthermore, that all such sets of solutions lie in bounded subsets of E^N and E^M , respectively. Then there exist bounded functions $\bar{z} \in Z_L$ and $\bar{w} \in W_L$ such that $\int_0^T \bar{z}(t) \cdot a(t) dt = \int_0^T \bar{w}(t) \cdot c(t) dt$.

PROOF. For the step function \hat{z}^n associated to $\{z^n(t_0^n), \dots, z^n(t_n^n)\}$ write $\hat{z}^n(t) = (\zeta_1^n(t), \dots, \zeta_N^n(t))$, $0 \leq t \leq T$. By hypothesis there exists $\rho > 0$ such that $|\zeta_j^n(t)| \leq \rho$ for $j = 1, \dots, N$; $n = 1, 2, \dots$; and $0 \leq t \leq T$, so $\zeta_j^n \in L^\infty[0, T]$ for $j = 1, \dots, N$; $n = 1, 2, \dots$, and $\|\zeta_j^n\|_\infty \leq \rho$.

We use a diagonal process to find a weak* limit. By Lemma 3 there exist $\bar{\zeta}_1 \in L^\infty[0, T]$ and a subsequence $\{n_k\}$ such that

$$\int_{t_1}^{t_2} \zeta_1^{n_k}(t) dt \longrightarrow \int_{t_1}^{t_2} \bar{\zeta}_1(t) dt$$

as $k \longrightarrow \infty$ for $0 \leq t_1 < t_2 \leq T$. Next, apply Lemma 3 to

$\{\xi_2^{n_k}\}$ to obtain a $\bar{\xi}_2 \in L^\infty[0, T]$ such that a subsequence of $\{\xi_2^{n_k}\}$ converges weak* to $\bar{\xi}_2$. Repeat taking successive subsequences to obtain $\bar{\xi}_j \in L^\infty[0, T]$ for $j = 1, \dots, N$, and a common subsequence $\{n_\ell\}$ such that

$$(8.1) \quad \int_{t_1}^{t_2} \xi_j^{n_\ell}(t) dt \longrightarrow \int_{t_1}^{t_2} \bar{\xi}_j(t) dt \quad \text{as } \ell \longrightarrow \infty,$$

for $j = 1, \dots, N$; $0 \leq t_1 < t_2 \leq T$.

Since $0 \leq \xi_j^{n_\ell}(t) \leq \rho$ for $t \in [0, T], j = 1, \dots, N$ and $\ell = 1, 2, \dots$, it follows immediately from (8.1) that $0 \leq \bar{z}(t) \leq \rho u$ for almost all $t \in [0, T]$, where $u = (1, 1, \dots) \in E^N$. By taking an equivalent representative in L_N^∞ we may assume that

$$(8.2) \quad 0 \leq \bar{z}(t) \leq \rho u, \quad 0 \leq t \leq T.$$

To prove $\bar{z} \in Z_L$ it is convenient to define a step function $\hat{c}^n : [0, T] \longrightarrow E^M$ associated to $\{c^n(t_0^n), \dots, c^n(t_n^n)\}$ by

$$\hat{c}^n(t) = \begin{cases} c^n(t_k^n), & t_k^n \leq t < t_{k+1}^n, \quad k = 0, \dots, n-1 \\ c^n(t_n^n), & t = t_n^n. \end{cases}$$

We now assert: For all n and for $t, 0 \leq t < T$,

there exists $x_n(t) \in E^M$ such that

$$(8.3) \quad \hat{Bz}^n(t) \leq \hat{c}^n(t) + \int_0^t \hat{Cz}^n(s)ds + x_n(t)$$

where $\lim_n x_n(t) = 0$, uniformly in t for $t \in [0, T)$.

Now given any n and $t \in [0, T)$, there exists a unique k , $0 \leq k \leq n-1$, such that $t_k^n \leq t < t_{k+1}^n$. By definition $\hat{z}^n(t) = z^n(t_k^n)$ and $\hat{c}^n(t) = c^n(t_k^n)$. Also by definition

$$\int_0^t \hat{Cz}^n(s)ds = \Delta t^n \sum_{v=0}^{k-1} Cz^n(t_v^n) + (t - t_k^n)Cz^n(t_k^n)$$

where we agree that $\sum_{v=0}^{-1} z^n(t_v^n) = 0$ should $k = 0$.

It is clear that this procedure uniquely defines the term $(t - t_k^n)Cz^n(t_k^n)$ (call it $-x_n(t)$) for any n and $t \in [0, T)$ provided k is such that $t_k^n \leq t < t_{k+1}^n$. Note that (for any linear norm on E^M) $\|x_n(t)\| \leq \Delta t^n \|Cz^n(t_k^n)\|$ for all n and $t \in [0, T)$. Now by hypothesis there exists $\rho > 0$ such that $0 \leq z^n(t_k^n) \leq \rho u$, for all n and k , $0 \leq k \leq n$. Hence there exists $\rho' > 0$ such that $\|Cz^n(t_k^n)\| \leq \rho'$ for all n and k , $0 \leq k \leq n$. Thus for all n and for $t \in [0, T)$, $\|x_n(t)\| \leq T/n \rho'$, so that $\lim_n x_n(t) = 0$, uniformly in t for $t \in [0, T)$. The inequality (8.3) now follows from the fact that the $z^n(t_v^n)$ are feasible for P^n . In particular

$$(8.4) \quad Bz^n(t_k^n) \leq c^n(t_k^n) + \Delta t^n \sum_{v=0}^{k-1} Cz^n(t_v^n),$$

so that $\hat{Bz}^n(t) = \int_0^t \hat{Cz}^n(s)ds \leq \hat{c}^n(t) + x_n(t)$. This proves the assertion.

Now integrating (8.3) from t_1 to t_2 , $0 \leq t_1 < t_2 \leq T$, we get, for $n = 1, 2, \dots$

$$(8.5) \quad \int_{t_1}^{t_2} \hat{Bz}^n(t)dt \leq \int_{t_1}^{t_2} \hat{c}^n(t)dt \\ + \int_{t_1}^{t_2} \left(\int_0^t \hat{Cz}^n(s)ds \right)dt + \int_{t_1}^{t_2} x_n(t)dt.$$

But by (8.1), p. 25,

$$\int_{t_1}^{t_2} \hat{Bz}^{n_l}(t)dt \longrightarrow \int_{t_1}^{t_2} B\bar{z}(t)dt$$

and

$$\int_{t_1}^{t_2} \left(\int_0^t \hat{Cz}^{n_l}(s)ds \right)dt \longrightarrow \int_{t_1}^{t_2} \left(\int_0^t C\bar{z}(s)ds \right)dt$$

as $l \longrightarrow \infty$ for $0 \leq t_1 < t_2 \leq T$. Now if

$$\int_{t_1}^{t_2} \hat{c}^{n\ell}(t) dt \longrightarrow \int_{t_1}^{t_2} c(t) dt$$

as $\ell \longrightarrow \infty$, for $0 \leq t_1 < t_2 \leq T$, then it follows from (8.5) and these results that $\bar{z} \in Z_L^1$, and hence $\bar{z} \in Z_L$.

We prove that

$$\int_{t_1}^{t_2} \hat{c}^{n\ell}(t) dt \longrightarrow \int_{t_1}^{t_2} c(t) dt$$

by proving that for $i = 1, \dots, M$,

$$\int_{t_1}^{t_2} \hat{\gamma}_1^n(t) dt \longrightarrow \int_{t_1}^{t_2} \gamma_1(t) dt ,$$

$0 \leq t_1 < t_2 \leq T$, as $n \longrightarrow \infty$. Let

$$\mu = \max \{ |\gamma_1(t)| : i = 1, \dots, M, 0 \leq t \leq T \}$$

$$\geq \sup \{ |\hat{\gamma}_1^n(t_\nu^n)| : i = 1, \dots, M, n = 1, 2, \dots; 0 \leq \nu \leq n \} .$$

Now it is clear that for any i $|\hat{\gamma}_1^n(t)| \leq \mu$ for $0 \leq t \leq T$, $n = 1, 2, \dots$. Thus it suffices to prove that $\hat{\gamma}_1^n(t) \longrightarrow \gamma_1(t)$ almost everywhere, since the desired result then follows from the Lebesgue dominated convergence theorem.

If $t_0 \in (0, T)$ then there is a unique interval $[t_{\nu(n)}^n, t_{\nu(n)+1}^n)$ containing t_0 . Consider i fixed,

$1 \leq i \leq M$. By definition and the law of the mean, for each $n > T/t_0$ there exists τ_n , $t_{v(n)-1}^n \leq \tau_n \leq t_{v(n)}^n$, such that

$$\hat{\gamma}_1^n(t_0) = \frac{1}{\Delta t^n} \int_{t_{v(n)-1}^n}^{t_{v(n)}^n} \gamma_1(t) dt = \gamma_1(\tau_n) \quad .$$

But it is clear that $\tau_n \longrightarrow t_0$ as $n \longrightarrow \infty$, so that $\gamma_1(\tau_n) \longrightarrow \gamma_1(t_0)$, giving $\hat{\gamma}_1^n(t_0) \longrightarrow \gamma_1(t_0)$ for $t_0 \in (0, T)$. This was to be shown.

To obtain the desired $\bar{w} \in W_L$ look at the sequence $\{\hat{w}^{n_\ell}\}$, where $\xi_j^{n_\ell} \longrightarrow \bar{\xi}_j$ (weak*) for $j = 1, \dots, N$, as $\ell \longrightarrow \infty$ ((8.1), p. 25). By a diagonal process similar to that used previously there exists a function $\bar{w} \in L_M^\infty$ and a subsequence of $\{n_\ell\}$, say $\{n_r\}$, such that $\omega_i^{n_r} \longrightarrow \bar{\omega}_i$ (weak*) as $r \longrightarrow \infty$ for $i = 1, \dots, M$.

By an argument analogous to the previous one, we get $\bar{w} \in W_L$, and it may be assumed that for some positive $\bar{\rho}$

$$0 \leq \bar{w}(t) \leq \bar{\rho}v, \quad 0 \leq t \leq T, \quad (8.6)$$

where $v = (1, 1, \dots, 1) \in E^M$.

To complete the proof of the lemma it remains to prove that $\int_0^T \bar{z}(t) \cdot a(t) dt = \int_0^T \bar{w}(t) \cdot c(t) dt$.

Now using the duality theorem of linear programming and the hypothesis that for $n = 1, 2, \dots$ the vectors $z^n(t_v^n)$

and $w^n(t_v^n)$ solve P^n , we get

$$(8.7) \quad \sum_{v=0}^n z^n(t_v^n) \cdot a^n(t_v^n) = \sum_{v=0}^n w^n(t_v^n) \cdot c^n(t_v^n), \quad n = 1, 2, \dots$$

Note that for $v = 0, \dots, n-1$, if $t_v^n \leq t < t_{v+1}^n$, $\hat{z}^n(t) = z^n(t_v^n)$. Hence, by the definitions

$$\begin{aligned} \Delta t^n \sum_{v=0}^{n-1} z^n(t_v^n) \cdot a^n(t_v^n) &= \sum_{v=0}^{n-1} z^n(t_v^n) \cdot \int_{t_v^n}^{t_{v+1}^n} a(t) dt \\ &= \sum_{v=0}^{n-1} \int_{t_v^n}^{t_{v+1}^n} \hat{z}^n(t) \cdot a(t) dt \\ &= \int_0^T \hat{z}^n(t) \cdot a(t) dt \end{aligned}$$

Similarly, for $v = 1, \dots, n$, if $t_{v-1}^n < t \leq t_v^n$, $\hat{w}^n(t) = w^n(t_v^n)$, so that

$$\begin{aligned} \Delta t^n \sum_{v=1}^n w^n(t_v^n) \cdot c^n(t_v^n) &= \sum_{v=1}^n w^n(t_v^n) \cdot \int_{t_{v-1}^n}^{t_v^n} c(t) dt \\ &= \sum_{v=1}^n \int_{t_{v-1}^n}^{t_v^n} \hat{w}^n(t) \cdot c(t) dt = \int_0^T \hat{w}^n(t) \cdot c(t) dt \end{aligned}$$

Thus, from (8.7), for $n = 1, 2, \dots$

$$\begin{aligned}
 (8.8) \quad & \int_0^T \hat{z}^n(t) \cdot a(t) dt + \Delta t^n z^n(t_n^n) \cdot a^n(t_n^n) \\
 &= \int_0^T \hat{w}^n(t) \cdot c(t) dt + \Delta t^n w^n(t_0^n) \cdot c^n(t_0^n)
 \end{aligned}$$

It is clear, however, that the vectors $z^n(t_n^n)$, $a^n(t_n^n)$, $w^n(t_0^n)$, and $c^n(t_0^n)$ remain bounded for all n so that

$$\lim_n \Delta t^n z^n(t_n^n) \cdot a^n(t_n^n) = \lim_n \Delta t^n w^n(t_0^n) \cdot c^n(t_0^n) = 0$$

We now use the fact that there is a common subsequence $\{n_r\}$ such that $\xi_j^{n_r} \longrightarrow \bar{\xi}_j$ and $\omega_1^{n_r} \longrightarrow \bar{\omega}_1$ (weak*) as $r \longrightarrow \infty$ for $i = 1, \dots, M$, $j = 1, \dots, N$. Thus, as $r \longrightarrow \infty$,

$$\int_0^T \hat{z}^{n_r}(t) \cdot a(t) dt \longrightarrow \int_0^T \bar{z}(t) \cdot a(t) dt, \text{ since } a \in L_N^1$$

and

$$\int_0^T \hat{w}^{n_r}(t) \cdot c(t) dt \longrightarrow \int_0^T \bar{w}(t) \cdot c(t) dt, \text{ since } c \in L_M^1.$$

But these limiting values $\int_0^T \bar{z}(t) \cdot a(t) dt$ and $\int_0^T \bar{w}(t) \cdot c(t) dt$ must be equal. This fact follows from (8.8) and the observation that the terms involving Δt^n

tend to zero.

This completes the proof of the main lemma.

We note that such functions \bar{z} and \bar{w} may not be solutions to the original problem, since they satisfy the inequalities only almost everywhere.

LEMMA 5 (Patch-up process). Given $z \in Z_L$ with $0 \leq z(t) \leq pu$ for $0 \leq t \leq T$, there exists a $\tilde{z} \in Z$ such that $\tilde{z} = z$ almost everywhere. (A similar result holds for the dual problem.)

PROOF. Let $S = \{t \in [0, T] : Bz(t) \leq c(t) + \int_0^t Cz(s)ds\}$. Then, by the hypothesis, if \tilde{S} denotes the complement of S in $[0, T]$, \tilde{S} has Lebesgue measure zero. Thus S is dense in $[0, T]$.

By the axiom of choice we wish to choose some bounded, measureable, non-negative function \tilde{z} such that $\tilde{z}(t) = z(t)$ for $t \in S$ and such that

$$(8.9) \quad B\tilde{z}(t) \leq c(t) + \int_0^t C\tilde{z}(s)ds, \quad 0 \leq t \leq T$$

Instead of (8.9) it suffices to have

$$(8.10) \quad B\tilde{z}(t) \leq c(t) + \int_0^t Cz(s)ds, \quad 0 \leq t \leq T$$

since \tilde{z} is to be equal to z almost everywhere, that is, for $t \in S$.

To prove the existence of such a function it suffices to prove that for each $t \in \tilde{S}$ there exists some $x \in E^N$ such that $Bx \leq c(t) + \int_0^t Cz(s)ds$ and $0 \leq x \leq \rho u$. Accordingly, let $t_0 \in \tilde{S}$. Since S is dense in $[0, T]$, for $k = 1, 2, \dots$ there exist $t_k \in S$ with $t_k \rightarrow t_0$ as $k \rightarrow \infty$. Now by the hypothesis $0 \leq z(t_k) \leq \rho u$ for $k = 1, 2, \dots$, so by compactness there exist $x \in E^N$ and a subsequence $\{k_\ell\}$ such that $z(t_{k_\ell}) \rightarrow x$. But for each ℓ , $Bz(t_{k_\ell}) \leq c(t_{k_\ell}) + \int_0^{t_{k_\ell}} Cz(s)ds$ so that in the limit $Bx \leq c(t_0) + \int_0^{t_0} Cz(s)ds$. This uses the continuity of c and of the integral. Furthermore, for $\ell = 1, 2, \dots$, $0 \leq z(t_{k_\ell}) \leq \rho u$, so that $0 \leq x \leq \rho u$.

Therefore, by the axiom of choice there exists some \tilde{z} having the desired properties.

9. BOUNDEDNESS FOR THE PRIMAL PROBLEM

Having proved the main Lemma 4 we shall seek to show that hypotheses I and II of Theorem 1, p. 4, are sufficient to reduce the theorem to a point where Lemma 4 may be applied.

In this section we show that Hypothesis I guarantees boundedness for the primal problems.

In E^N we know that if $\{x \in E^N : Ax \leq b\}$ is not empty, then

$$(9.1) \quad \begin{aligned} &\{x \in E^N : Ax \leq b\} \text{ is bounded if and only if} \\ &\{x \in E^N : Ax \leq 0\} = \{0\}. \end{aligned}$$

(A proof is given in [7].) We now prove

LEMMA 6. If $\{x \in E^N : x \geq 0 \text{ and } Ax \leq b\}$ is not empty, then $\{x \in E^N : x \geq 0 \text{ and } Ax \leq b\}$ is bounded if and only if $\{x \in E^N : x \geq 0 \text{ and } Ax \leq 0\} = \{0\}$.

PROOF. Let \hat{A} be the $(M+N) \times N$ matrix with rows \hat{a}_i defined by $\hat{a}_i = a_i, i = 1, \dots, M, \hat{a}_{M+i} = -(\delta_{ij}) \in E^N, i = 1, \dots, N$, where δ_{ij} is the Kronecker delta. Let $\hat{b} = (\hat{\beta}_i) \in E^{M+N}$ be defined by $\hat{\beta}_i = \beta_i, i = 1, \dots, M, \hat{\beta}_i = 0, i = M+1, \dots, M+N$. Then $\hat{A}x \leq \hat{b}$ if and only if $Ax \leq b$ and $x \geq 0$. Also, $\hat{A}x \leq 0$ if and only if $Ax \leq 0$ and $x \geq 0$. Thus Lemma 6 follows immediately from (9.1).

We will find it useful to define the following linear norm on E^N .

DEFINITION. For $x = (x_j) \in E^N$, let $\|x\| = \sum_{j=1}^N |x_j|$.

It is clear that a subset of E^N is bounded with respect to this norm if and only if it is bounded in the euclidean norm. Furthermore, the topologies induced on E^N by these norms are identical.

LEMMA 7. Suppose $\{z \in E^N : Az \leq 0\} = \{0\}$. Then there exists $\rho > 0$ such that $Az \leq x$ implies $\|z\| \leq \rho \|x\|$, whenever $z \in E^N$ and $x \in E^M$.

PROOF. Let $S_M = \{y \in E^M : \|y\| = 1\}$. Note that S_M

is compact. Now for $y \in S_M$ the set $\{z \in E^N : Az \leq y\}$ is compact, for it is either empty or bounded and closed by the hypothesis and (9.1). Define $\phi : S_M \longrightarrow \text{reals}$ by

$$\phi(y) = \begin{cases} \max \{ \|z\| : Az \leq y \}, & \text{if not empty} \\ 0, & \text{otherwise.} \end{cases}$$

We assert that ϕ is bounded on S_M , for suppose not. Then there exist $y_1, y_2, \dots \in S_M$ such that $\phi(y_n) \longrightarrow \infty$. We may assume that $\phi(y_n) > 0$ for each n . Hence for each n , $\{z : Az \leq y_n\}$ is not empty. Let z_n satisfy $\phi(y_n) = \|z_n\|$ and $Az_n \leq y_n$ for $n = 1, 2, \dots$. Define $\hat{z}_n = z_n / \|z_n\|$. Note that $\|\hat{z}_n\| = 1$ for all n . Hence $\hat{z}_n \in S_N$, so by compactness there exist $\bar{z} \in S_N$ and a subsequence $\{n_k\}$ such that $\hat{z}_{n_k} \longrightarrow \bar{z}$ as $k \longrightarrow \infty$. Now for $k = 1, 2, \dots$ $A\hat{z}_{n_k} \leq y_{n_k} / \|z_{n_k}\|$; but since $y_{n_k} \in S_M$, $y_{n_k} / \|z_{n_k}\| \longrightarrow 0$. Hence in the limit $A\bar{z} \leq 0$. But $\bar{z} \in S_N$ so $\bar{z} \neq 0$. This contradicts the hypothesis. Hence there exists $\rho > 0$ such that $0 \leq \phi(y) \leq \rho$ for $y \in S_M$.

Now suppose $Az \leq x$. If $\|x\| = 0$, $x = 0$, so $z = 0$ and clearly $\|z\| \leq \rho\|x\|$. If $\|x\| > 0$, then $x' = x / \|x\| \in S_M$. Hence $Az / \|x\| \leq x / \|x\|$ so $\|z / \|x\|\| \leq \phi(x') \leq \rho$. This completes the proof.

LEMMA 8. Suppose $\{z \in E^N : z \geq 0 \text{ and } Az \leq 0\} = \{0\}$. Then there exists $\rho > 0$ such that $z \geq 0$ and $Az \leq x$ implies $\|z\| \leq \rho\|x\|$ whenever $z \in E^N$ and $x \in E^M$.

PROOF. Lemma 8 follows immediately from Lemma 7 using the device employed in the proof of Lemma 6, p. 34.

LEMMA 9. Assume that $\{z \in E^N : z \geq 0 \text{ and } Bz \leq 0\} = \{0\}$. For $n = 1, 2, \dots$ let $z^n(t_0^n), \dots, z^n(t_n^n)$ be feasible for P^n . Then there exists $R > 0$ such that for all n and k , $0 \leq k \leq n$, $\|z^n(t_k^n)\| \leq R$.

PROOF. Recall (p. 28) that $\mu = \max \{|\gamma_i(t)| : i = 1, \dots, M, 0 \leq t \leq T\}$ so that for all n and k , $0 \leq k \leq n$, $c^n(t_k^n) \leq \mu v$.

Let an arbitrary n be given. By the hypothesis $z^n(t_0^n) \geq 0$ and $Bz^n(t_0^n) \leq c^n(t_0^n) \leq \mu v$, so by Lemma 8, $\|z^n(t_0^n)\| \leq \rho \|\mu v\|$. Also $z^n(t_1^n) \geq 0$ and $Bz^n(t_1^n) \leq c^n(t_1^n) + \Delta t^n C z^n(t_0^n) \leq \mu v + \Delta t^n C z^n(t_0^n)$ so

$$(9.2) \quad \|z^n(t_1^n)\| \leq \rho \|\mu v + \Delta t^n C z^n(t_0^n)\|.$$

We now claim

$$(9.3) \quad \text{There exists } \theta \geq 0 \text{ such that for } z \in E^N \\ \|Cz\| \leq \theta \|z\|.$$

For the proof let C be the $M \times N$ matrix (γ_{ij}) , and let $\theta/M = \max_{i,j} |\gamma_{ij}|$. Now $Cz = (c_1 \cdot z, \dots, c_M \cdot z) \in E^M$, where c_i is the i^{th} row of C . Note that $|c_i \cdot z| = |\sum_j \gamma_{ij} z_j| \leq \theta/M \sum_j |z_j| = \theta/M \|z\|$ for $i = 1, \dots, M$. Hence $\|Cz\| = \sum_{i=1}^M |c_i \cdot z| \leq \sum_{i=1}^M \theta/M \|z\| = \theta \|z\|$, proving the assertion.

Using this result in (9.2) we get

$$\begin{aligned}
 \|z^n(t_1^n)\| &\leq \rho\{\|\mu v\| + \Delta t^n \theta \|z^n(t_0^n)\|\} \\
 &\leq \rho\{\|\mu v\| + \Delta t^n \theta \|\mu v\|\} \\
 &= \rho\|\mu v\| (1 + \Delta t^n \theta \rho) .
 \end{aligned}$$

We now assert: For $0 \leq k \leq n$, $\|z^n(t_k^n)\| \leq \rho\|\mu v\| (1 + \Delta t^n \theta \rho)^k$. This is true for $k = 0, 1$. Assume true for all v , $0 \leq v \leq k < n$. Now $z^n(t_{k+1}^n) \geq 0$ and $Bz^n(t_{k+1}^n) \leq c^n(t_{k+1}^n) + \Delta t^n \sum_{v=0}^k Cz^n(t_v^n) \leq \mu v + \Delta t^n \sum_{v=0}^k Cz^n(t_v^n)$, so

$$\begin{aligned}
 \|z^n(t_{k+1}^n)\| &\leq \rho\left\{\|\mu v\| + \Delta t^n \sum_{v=0}^k \|Cz^n(t_v^n)\|\right\} \\
 &\leq \rho\left\{\|\mu v\| + \Delta t^n \theta \sum_{v=0}^k \|z^n(t_v^n)\|\right\} \\
 &\leq \rho\left\{\|\mu v\| + \Delta t^n \theta \rho \|\mu v\| \sum_{v=0}^k (1 + \Delta t^n \theta \rho)^v\right\} \\
 &= \rho\|\mu v\| (1 + \Delta t^n \theta \rho)^{k+1} .
 \end{aligned}$$

The assertion thus follows by induction on k .

Now $\theta \rho \geq 0$ and $\Delta t^n = T/n$, so $(1 + T\theta\rho/n)^k \leq (1 + T\theta\rho/n)^n$ for $0 \leq k \leq n$. Hence for $0 \leq k \leq n$, $\|z^n(t_k^n)\| \leq \rho\|\mu v\| (1 + T\theta\rho/n)^n$. But $\{(1 + T\theta\rho/n)^n\}_n$ is

an increasing sequence ([8], p. 72) and $\lim_n (1 + T\theta\rho/n)^n = \exp T\theta\rho$. Therefore for $n = 1, 2, \dots$, $0 \leq k \leq n$, $\|z^n(t_k^n)\| \leq \rho \|\mu v\| \exp T\theta\rho$. If this upper bound is called R it is clear that R is independent of n and k . This proves the lemma.

10. EFFECTIVE BOUNDEDNESS FOR THE DUAL PROBLEM

We turn now to the dual problem using Hypothesis II.

LEMMA 10. Assume that B, C , and $c(t)$ have non-negative components, $0 \leq t \leq T$. Then there exists $\rho > 0$ such that if the dual of P^n has a solution it has solution vectors with no component bigger than ρ .

PROOF. Define $\bar{\alpha} = \max \{|\alpha_j(t)| : j = 1, \dots, N, 0 \leq t \leq T\}$. Note that $\bar{\alpha} \geq \alpha_j^n(t_k^n)$ for $j = 1, \dots, N$, and all n and k , $0 \leq k \leq n$. Let $\bar{\beta} = \min \{\beta_{1j} : \beta_{1j} > 0\}$ and $\bar{\gamma} = \max_j \{\sum_1 \gamma_{1j}\}$, where $B = (\beta_{1j})$ and $C = (\gamma_{1j})$. Note that $\bar{\beta}, \bar{\gamma} > 0$, since we have assumed in the definition of the problem that neither B nor C is a zero matrix.

Now for $n = 1, 2, \dots$ define $\bar{w}^n(t_k^n)$ by $\bar{w}^n(t_k^n) = \bar{w}^n(t_k^n)v$, $0 \leq k \leq n$, where $\bar{w}^n(t_k^n) = \bar{\alpha}/\bar{\beta} (1 + \bar{\gamma}/\bar{\beta} \Delta t^n)^{n-k}$. Note that for $0 \leq k \leq n$, $(1 + \bar{\gamma}/\bar{\beta} \Delta t^n)^{n-k} \leq (1 + \bar{\gamma}/\bar{\beta} \Delta t^n)^n$ and $(1 + \bar{\gamma}/\bar{\beta} \Delta t^n)^n \uparrow \exp \bar{\gamma}T/\bar{\beta}$ as $n \rightarrow \infty$. Therefore, $\bar{w}^n(t_k^n) \leq \bar{\alpha}/\bar{\beta} \exp \bar{\gamma}T/\bar{\beta}$ for $n = 1, 2, \dots$, $0 \leq k \leq n$. This bound will be the desired ρ .

Now consider n fixed and let $w^n(t_0^n), \dots, w^n(t_n^n)$

be feasible for the dual of P^n . The argument to follow will hold independently of n .

Define $\hat{w}^n(t_k^n) = (\hat{w}_1^n(t_k^n), \dots, \hat{w}_M^n(t_k^n)) \in E^M$ by
 $\hat{w}_i^n(t_k^n) = \min \{\omega_i^n(t_k^n), \bar{w}^n(t_k^n)\}$ for $i = 1, \dots, M$;
 $k = 0, \dots, n$. Clearly $\hat{w}^n(t_k^n) \leq w^n(t_k^n)$, and $\hat{w}^n(t_k^n) \leq \bar{w}^n(t_k^n)$, and $\hat{w}^n(t_k^n) \geq 0$ for $k = 0, \dots, n$.

We will show that $\hat{w}^n(t_0^n), \dots, \hat{w}^n(t_n^n)$ is feasible for the dual of P^n , so that if $w^n(t_0^n), \dots, w^n(t_n^n)$ is optimal for P^n ,

$$\sum_{k=0}^n \hat{w}^n(t_k^n) \cdot c^n(t_k^n) \leq \sum_{k=0}^n w^n(t_k^n) \cdot c^n(t_k^n)$$

and we see that $\hat{w}^n(t_0^n), \dots, \hat{w}^n(t_n^n)$ is also optimal.

Furthermore, since $\hat{w}^n(t_k^n) \leq \bar{w}^n(t_k^n)$ for $k = 0, \dots, n$, each component of $\hat{w}^n(t_k^n)$ is no more than $\bar{\alpha}/\bar{\beta} \exp \bar{\gamma}T/\bar{\beta} \equiv \rho$.

It remains to prove that $\hat{w}^n(t_0^n), \dots, \hat{w}^n(t_n^n)$ is feasible for the dual of P^n , that is

$$\hat{w}^n(t_n^n)_B \geq a^n(t_n^n)$$

and

$$\hat{w}^n(t_k^n)_B \geq a^n(t_k^n) + \Delta t^n \sum_{v=k+1}^n \hat{w}^n(t_v^n)_C, \quad k = 0, \dots, n-1.$$

In the sequel fix j at an arbitrary value $1 \leq j \leq N$. The argument will be independent of j .

Firstly, for $k = n$ we must show $\sum_{i=1}^M \hat{\omega}_i^n(t_n^n) \beta_{ij} \geq \alpha_j^n(t_n^n)$.

CASE 1. There exists i_0 such that $\beta_{i_0 j} > 0$ and

$\hat{\omega}_{i_0}^n(t_n^n) = \bar{\omega}^n(t_n^n)$. Then $\sum_i \hat{\omega}_i^n(t_n^n) \beta_{ij} \geq \hat{\omega}_{i_0}^n(t_n^n) \beta_{i_0 j} = \bar{\omega}^n(t_n^n) \beta_{i_0 j} = \beta_{i_0 j} \bar{\alpha} / \bar{\beta} \geq \bar{\alpha} \geq \alpha_j^n(t_n^n)$, since $\beta_{i_0 j} > 0$ implies $\beta_{i_0 j} / \bar{\beta} \geq 1$, and $\bar{\alpha} \geq \alpha_j^n(t_n^n)$ independent of j .

CASE 2. Whenever $\beta_{ij} > 0$, $\hat{\omega}_i^n(t_n^n) = \omega_i^n(t_n^n)$. Then $\sum_i \hat{\omega}_i^n(t_n^n) \beta_{ij} = \sum_i \omega_i^n(t_n^n) \beta_{ij} \geq \alpha_j^n(t_n^n)$ since $\{w^n(t_0^n), \dots, w^n(t_n^n)\}$ is feasible for P^n .

It remains to show for $k = 0, \dots, n-1$

$$\sum_{i=1}^M \hat{\omega}_i^n(t_k^n) \beta_{ij} \geq \alpha_j^n(t_k^n) + \Delta t^n \sum_{v=k+1}^n \left(\sum_{i=1}^M \gamma_{ij} \hat{\omega}_i^n(t_v^n) \right)$$

Consider k fixed, $0 \leq k \leq n-1$.

CASE 1. There exists i_0 such that $\beta_{i_0 j} > 0$ and

$\hat{\omega}_{i_0}^n(t_k^n) = \bar{\omega}^n(t_k^n)$. Then $\sum_i \hat{\omega}_i^n(t_k^n) \beta_{ij} \geq \beta_{i_0 j} \hat{\omega}_{i_0}^n(t_k^n) = \beta_{i_0 j} \bar{\omega}^n(t_k^n) = \beta_{i_0 j} \bar{\alpha} / \bar{\beta} (1 + \bar{\gamma} / \bar{\beta} \Delta t^n)^{n-k} \geq \bar{\alpha} (1 + \bar{\gamma} / \bar{\beta} \Delta t^n)^{n-k}$.

Now since $\hat{w}^n(t_v^n) \leq \bar{w}^n(t_v^n)$ for $v = 0, \dots, n$, $\sum_i \gamma_{ij} \hat{\omega}_i^n(t_v^n) \leq \sum_i \gamma_{ij} \bar{\omega}^n(t_v^n) \leq \bar{\gamma} \bar{\omega}^n(t_v^n)$. Therefore

$$\begin{aligned}
\alpha_j^n(t_k^n) + \Delta t^n \sum_{v=k+1}^n \left(\sum_{i=1}^M \gamma_{ij} \hat{\omega}_i^n(t_v^n) \right) &\leq \bar{\alpha} + \Delta t^n \sum_{v=k+1}^n \bar{\gamma} \bar{\omega}^n(t_v^n) \\
&= \bar{\alpha} + \Delta t^n \bar{\gamma} \bar{\alpha} / \bar{\beta} \sum_{v=k+1}^n (1 + \bar{\gamma} / \bar{\beta} \Delta t^n)^{n-v} \\
&= \bar{\alpha} \left(1 + \bar{\gamma} / \bar{\beta} \Delta t^n \sum_{r=0}^{n-k-1} (1 + \bar{\gamma} / \bar{\beta} \Delta t^n)^r \right) \\
&= \bar{\alpha} (1 + \bar{\gamma} / \bar{\beta} \Delta t^n)^{n-k}
\end{aligned}$$

which was to be shown.

CASE 2. Whenever $\beta_{ij} > 0$, $\hat{\omega}_i^n(t_k^n) = \omega_i^n(t_k^n)$.

Then

$$\begin{aligned}
\sum_i \hat{\omega}_i^n(t_k^n) \beta_{ij} &= \sum_i \omega_i^n(t_k^n) \beta_{ij} \geq \alpha_j^n(t_k^n) \\
&+ \Delta t^n \sum_{v=k+1}^n w^n(t_v^n) C \\
&\geq \alpha_j^n(t_k^n) + \Delta t^n \sum_{v=k+1}^n \hat{w}^n(t_v^n) C
\end{aligned}$$

since $w^n(t_v^n) \geq \hat{w}^n(t_v^n)$ for $v = 0, \dots, n$, and $C \geq 0$.

This proves Lemma 10.

REMARK. It is easily seen that if $B \geq 0$ then

$\{x \in E^N : x \geq 0 \text{ and } Bx \leq 0\} = \{0\}$ if and only if each

column of B has a positive entry.

Lemma 10 has an immediate corollary.

LEMMA 11. If the hypotheses of Lemma 10 are satisfied and if each column of B has a positive entry, then the $\bar{w}^n(t_0^n), \dots, \bar{w}^n(t_n^n)$ are feasible for the dual of P^n , $n = 1, 2, \dots$.

PROOF. We are in Case 1 of the proof of Lemma 10.

11. PROOF OF THE DUALITY THEOREM

After proving Lemma 12 we shall be in a position to prove Theorem 1.

LEMMA 12. Under Hypotheses I and II of Theorem 1, for $n = 1, 2, \dots$ each P^n has optimizing solutions $z^n(t_k^n), w^n(t_k^n), k = 0, \dots, n$.

PROOF. For any positive integer n we need only show that both P^n and its dual problem have feasible vectors, since the duality theorem of linear programming will then yield the existence of optimizing solutions.

We note that $c^n(t_k^n) \geq 0$ for $k = 0, \dots, n$ so that $z^n(t_k^n) = 0, k = 0, \dots, n$, is feasible for P^n . By Lemma 11 and the remark preceding that lemma there are vectors feasible for the dual of P^n . This completes the proof.

We now prove Theorem 1. Lemma 12 applies, so for $n = 1, 2, \dots$ let $z^n(t_k^n), w^n(t_k^n), k = 0, \dots, n$ be the optimizing solutions of P^n . By Lemma 9, p. 36, there

exists $R > 0$ such that for $n = 1, 2, \dots$ and $0 \leq k \leq n$, $\|z^n(t_k^n)\| \leq R$. Furthermore, by Lemma 10, p. 38, we may assume that for some $\bar{\rho} > 0$, $\|w^n(t_k^n)\| \leq \bar{\rho}$ for $n = 1, 2, \dots$ and $0 \leq k \leq n$.

We have now succeeded in satisfying the hypotheses of the main Lemma 4, p. 24. Thus there exist functions $\bar{z} \in Z_L$ and $\bar{w} \in W_L$ with $\int_0^T \bar{z}(t) \cdot a(t) dt = \int_0^T \bar{w}(t) \cdot c(t) dt$. By condition (8.2), p. 25, we may assume that $0 \leq \bar{z}(t) \leq Ru$, $0 \leq t \leq T$, since R is clearly adequate to play the role of ρ , which was an upper bound on each component of $z^n(t_k^n)$ for all n and k , $0 \leq k \leq n$. Similarly, by condition (8.6), p. 29, we may assume that $0 \leq \bar{w}(t) \leq \bar{\rho}v$, $0 \leq t \leq T$.

We may now use the patch-up process (Lemma 5, p. 32) and further assume that $\bar{z} \in Z$ and $\bar{w} \in W$.

We have thus completed the proof of Theorem 1, for by the optimality condition of Theorem 2 (p. 11) \bar{z} and \bar{w} are solutions of their respective problems.

12. RELATED RESULTS

We have seen that Hypothesis I was sufficient to give a uniform bound on the functions $\{\hat{z}^n\}$ for the primal problem P^n and that Hypothesis II guaranteed a uniformly bounded set of functions $\{\hat{w}^n\}$ for the dual of P^n .

That I is related to Z and II is related to W is

even more apparent in the following two theorems, which can be proved with a minimum of further work.

THEOREM 7. Under Hypothesis I, if Z is not empty, then there exists a $\bar{z} \in Z$ which maximizes $\int_0^T z(t) \cdot a(t) dt$, for $z \in Z$.

THEOREM 8. Under Hypothesis II, if W is not empty, then there exists a $\bar{w} \in W$ which minimizes $\int_0^T w(t) \cdot c(t) dt$, for $w \in W$.

We need

LEMMA 13. Let u and v be bounded, measurable, non-negative functions on $[0, T]$ satisfying $u(t) \leq A + \int_0^t u(s)v(s)ds$, $0 \leq t \leq T$, where $A > 0$. Then $u(t) \leq A \exp \int_0^t v(s)ds$, for $0 \leq t \leq T$.

For a proof, see [9], pp. 35-36.

We can now prove a continuous analogue of Lemma 9, p.36.

LEMMA 14. Under Hypothesis I Z is either bounded or empty.

PROOF. If $z \in Z$ let $x(t) = c(t) + \int_0^t Cz(s)ds$, $0 \leq t \leq T$. Then for $0 \leq t \leq T$

$$\begin{aligned} \|x(t)\| &= \sum_{i=1}^M \left| \gamma_i(t) + \int_0^t c_i \cdot z(s)ds \right| \\ &\leq M\mu + \sum_{i=1}^M \left| \int_0^t c_i \cdot z(s)ds \right| \\ &\leq M\mu + \int_0^t \|Cz(s)\|ds \leq M\mu + \theta \int_0^t \|z(s)\|ds \end{aligned}$$

where μ is defined on p. 28 and θ on p. 36, (9.3).

Now by Lemma 8, p. 35, there exists a $\rho > 0$ such that $z(t) \geq 0$ and $Bz(t) \leq x(t)$ implies $\|z(t)\| \leq \rho \|x(t)\|$, $0 \leq t \leq T$. In fact ρ is independent of z, x . Hence $\|z(t)\| \leq \rho(M\mu + 1) + \rho\theta \int_0^t \|z(s)\| ds$, $0 \leq t \leq T$. Note that $\rho(M\mu + 1) > 0$. Letting $u(t) = \|z(t)\|$ and $v(t) = \rho\theta$, we apply Lemma 13 to get for $0 \leq t \leq T$

$$\|z(t)\| \leq \rho(M\mu + 1) \exp \rho\theta \int_0^t ds \leq \rho(M\mu + 1) \exp \rho\theta T.$$

Since this bound is independent of z , the lemma is proved.

We now prove Theorem 7. Let $\{z^n\}$ be a maximizing sequence; that is $\lim_n \int_0^T z^n(t) \cdot a(t) dt = \sup \{ \int_0^T z(t) \cdot a(t) dt : z \in Z \}$, where $z^n \in Z$ for $n = 1, 2, \dots$. Let m be the value of this supremum. Since Z is bounded by Lemma 14 $m < \infty$, and we may apply a diagonal process to each sequence $\{\xi_j^n\}$ to obtain a $\bar{z} \in Z_L$ and some subsequence n_k such that $\xi_j^{n_k} \xrightarrow{*} \bar{\xi}_j$ (weak*) as $k \rightarrow \infty$, for $j = 1, \dots, N$. Hence $\int_0^T z^{n_k}(t) \cdot a(t) dt \rightarrow \int_0^T \bar{z}(t) \cdot a(t) dt = m$. By the patch-up process, Lemma 5, p. 32, we may assume $\bar{z} \in Z$. This proves Theorem 7.

For the proof of Theorem 8 we need the continuous analogue of Lemma 10, p. 38.

LEMMA 15. Under Hypothesis II if $w \in W$ then there exists $\hat{w} \in W$ such that $\hat{\omega}_1(t) \leq \bar{\alpha}/\bar{\beta} \exp \bar{\gamma}T/\bar{\beta}$,
 $i = 1, \dots, M$ and $\int_0^T \hat{w}(t) \cdot c(t)dt \leq \int_0^T w(t) \cdot c(t)dt$.

PROOF. Analogous to the $\bar{w}^n(t_k^n)$ of Lemma 10 define $\bar{w}(t) = \bar{\alpha}/\bar{\beta} \exp \bar{\gamma}(T - t)/\bar{\beta}$ $v = \bar{w}(t)v$, for $0 \leq t \leq T$. For $w \in W$ define \hat{w} by $\hat{\omega}_1(t) = \min \{\omega_1(t), \bar{w}(t)\}$ for $i = 1, \dots, M$. The proof that $\hat{w} \in W$ is analogous to the proof that the $\hat{w}^n(t_0^n), \dots, \hat{w}^n(t_n^n)$ were feasible for the dual of P^n and is omitted. By the definition $\hat{w}(t) \leq w(t)$, $0 \leq t \leq T$, so that $\int_0^T \hat{w}(t) \cdot c(t)dt \leq \int_0^T w(t) \cdot c(t)dt$. Also $\omega_1(t) \leq \bar{w}(t) \leq \bar{\alpha}/\bar{\beta} \exp \bar{\gamma}T/\bar{\beta}$ for $0 \leq t \leq T$ and for $i = 1, \dots, M$. This proves the lemma.

For the proof of Theorem 8 let $\{w^n\}$ be a minimizing sequence. That is $\lim_n \int_0^T w^n(t) \cdot c(t)dt = \inf \{\int_0^T w(t) \cdot c(t)dt : w \in W\} \geq 0$. Let m' be the value of this infimum. By Lemma 15 we may assume that each $w^n \in W$ satisfies $\|w^n(t)\| < M \bar{\alpha}/\bar{\beta} \exp \bar{\gamma}T/\bar{\beta}$, for $0 \leq t \leq T$. Again using a diagonal process and the patch-up lemma, we find a function $w^* \in W$ and a subsequence $\{n_k\}$ such that $\int_0^T w^{n_k}(t) \cdot c(t)dt \longrightarrow \int_0^T w^*(t) \cdot c(t)dt = m'$. This $w^* \in W$ is the desired minimizing function, so the proof of Theorem 8 is complete.

13. AN ECONOMIC APPLICATION TO A DYNAMIC LEONTIEF MODEL

In view of the economic motivation of Theorem 1, one might well question whether Hypothesis I and, particularly, Hypothesis II are too restrictive for any fruitful economic application. In this section we describe a dynamic "closed-end" Leontief production model, one in which all goods are accumulated or consumed in the production system itself, with no flow of goods to or from the system. This model is based upon a continuous version of the discrete Leontief model discussed in [3], particularly on p. 289, with the additional requirement that there be no outside consumption. We shall see that Hypothesis II is satisfied for this model. Furthermore, Hypothesis I will seem not unreasonable.

Consider a production system consisting of N activities and N goods, where each activity produces exactly one good. Assume that units and notation are chosen so that operating the i^{th} activity at unit rate produces one unit of good G_i . Let α_{ij} be the amount of G_i consumed by the j^{th} activity in producing one unit of G_j , and let β_{ij} be the amount of G_i required as capital stock in order to produce G_j at a unit rate. Let $A = (\alpha_{ij})$ and $\bar{B} = (\beta_{ij})$, and note that the α_{ij}, β_{ij} are non-negative by definition.

If we assume that

(13.1) $x : [0, T] \longrightarrow E^N$ is a bounded, measurable function,

then the instantaneous net production resulting from operating the activities at a level $x(t) \geq 0$ is $(I - A)x(t)$. We impose the condition

$$(13.2) \quad (I - A)x(t) \geq 0, \quad 0 \leq t \leq T$$

which states that there should be no disinvestment (or decumulation) of stocks in the production process. (In [3] this condition is imposed on the model described on p. 289, but not imposed on the model described on p. 338.)

Now it is known (see, for example, [1], pp. 296-297) that a consumption matrix A is productive (meaning that some positive bill of goods $(I - A)x$ can be produced by some $x \geq 0$) if and only if $I - A$ has a non-negative inverse. We make the assumption that A is productive, so if $x(t)$ satisfies (13.2) $x(t) \geq 0$.

Now if c_0 denotes the initial stock bundle ($c_0 \in E^N$, $c_0 \geq 0$), then the stock bundle accumulated by time t is $c_0 + \int_0^t (I - A)x(s)ds$. The technological constraint imposed by limited capital stocks is, then,

$$(13.3) \quad \bar{E}x(t) \leq c_0 + \int_0^t (I - A)x(s)ds, \quad 0 \leq t \leq T.$$

Let $a : [0, T] \longrightarrow E^N$ be a continuous map and regard $a(t)$ as the value of the goods bundle $(1, 1, \dots, 1) \in E^N$

at time t . Denote by P the problem of maximizing the value $\int_0^T (I - A)x(t) \cdot a(t)dt$ subject to the constraints (13.1), (13.2), and (13.3), with the assumption that A is productive.

We have the following corollary to Theorem 1.

COROLLARY. If in the dynamic Leontief production model P the matrix A is productive and the matrix \bar{B} satisfies hypothesis I of Theorem 1, then the conclusion of Theorem 1 is valid for the model P .

PROOF. We employ a change of variable, letting $z(t) = (I - A)x(t)$. Since A is productive $x(t) = (I - A)^{-1}z(t)$, and (13.3) becomes

$$(13.4) \quad \bar{B}(I - A)^{-1}z(t) \leq c_0 + \int_0^t z(s)ds, \quad 0 \leq t \leq T.$$

Denote by P' the problem of finding some bounded, measurable, non-negative function $z : [0, T] \rightarrow E^N$ subject to (13.4) and maximizing $\int_0^T z(t) \cdot a(t)dt$. Since $z(t) \geq 0$ if x satisfies (13.2), and $x(t) \geq 0$ and x satisfies (13.2) if $z(t) \geq 0$, it is readily verified that the problems P and P' are equivalent. Furthermore, it is a straight-forward exercise to verify that the dual problems for P and P' also are equivalent, if one notes that the dual of the program P (with no sign restriction on x) requires strict equality in the dual constraints.

For the proof of the corollary we let $B = \bar{B}(I - A)^{-1}$,

$C = I$, and $c(t) \equiv c_0$ to employ the terms used in defining the continuous linear programming problems on p. 3 and verify that the hypotheses of Theorem 1 are satisfied. Since \bar{B} , $(I - A)^{-1}$, I , and c_0 all have non-negative components, Hypothesis II follows at once. Using the fact that $(I - A)^{-1}$ is both non-negative and non-singular, one easily verifies that B satisfies Hypothesis I if \bar{B} does, and this completes the proof.

14. BIBLIOGRAPHY

- [1] David Gale, "The Theory of Linear Economic Models," McGraw-Hill Book Company, Inc., New York, 1960.
- [2] Richard Bellman, "Dynamic Programming," Princeton University Press, Princeton, N. J., 1957.
- [3] R. Dorfman, P. A. Samuelson, and R. M. Solow, "Linear Programming and Economic Analysis," McGraw-Hill Book Company, Inc., New York, 1958.
- [4] Philip Wolfe, "A Dynamic Linear Programming Model," privately circulated.
- [5] R. Sherman Lehman, "On the Continuous Simplex Method," RAND Research Memorandum RM-1386, Santa Monica, Calif., 1954.
- [6] Stefan Banach, "Théorie des Opérations Linéaires," Subwencji funduszu kultury narodowej, Warszawa, 1932.
- [7] A. J. Goldman, "Resolution and Separation Theorems for Polyhedral Convex Sets," pp. 41-51 in H. W. Kuhn and A. W. Tucker, editors, "Linear Inequalities and Related Systems," Princeton University Press, Princeton, N. J., 1956.
- [8] Angus E. Taylor, "Advanced Calculus," Ginn and Company, Boston, 1955.
- [9] Richard Bellman, "Stability Theory of Differential Equations," McGraw-Hill Book Company, Inc., New York, 1953.